

MEM6810 Engineering Systems Modeling and Simulation



工程系统建模与仿真

Theory Analysis

Lecture 9: Output Analysis II: Comparison

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董浩云航运与物流研究院

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Sino-US Global Logistics Institute (Institute of Industrial & System Engineering)



- 1 Introduction
- 2 Comparison of Two Designs
 - ▶ Significant Difference
 - ▶ Independent Sampling
 - ▶ Common Random Numbers
- 3 Comparison of Multiple Designs
 - ▶ Bechhofer's Procedure
 - ▶ Paulson's Procedure
 - ▶ Ranking and Selection Review
 - ▶ Multi-Arm Bandit Problem



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- We now discuss how to compare two or more simulation models, i.e. to estimate their *relative performance*.
- Here, different simulation models may refer to different designs, operation policies, etc., of a simulated system; in this lecture we simply call them *different (system) designs*.
- It is one of the most important uses of simulation.

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 - the **actual differences** on the expected performance of system designs?
 - or the **random errors** in the simulation outputs?
- The comparison can be classified into two types:
 - Two system designs: using confidence interval of the difference.
 - Multiple (more than two) system designs: selection of the best.

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<i>System</i>	<i>Replication</i>				<i>Sample Mean</i>	<i>Sample Variance</i>
	<i>1</i>	<i>2</i>	\dots	<i>R_i</i>		
1	Y_{11}	Y_{21}	\dots	$Y_{R_1 1}$	\bar{Y}_1	S_1^2
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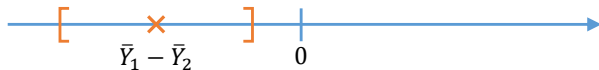
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- Point estimator of $\theta_1 - \theta_2$: $\bar{Y}_1 - \bar{Y}_2$.
- Approximate $1 - \alpha$ CI: $\bar{Y}_1 - \bar{Y}_2 \pm t_{v, 1-\alpha/2} \times \text{s.e.}(\bar{Y}_1 - \bar{Y}_2)$.
 - $\text{s.e.}(\bar{Y}_1 - \bar{Y}_2)$ is the estimator of standard error of $\bar{Y}_1 - \bar{Y}_2$; see more details about this quantity and v later.

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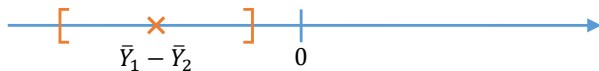
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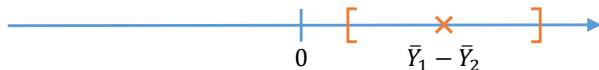


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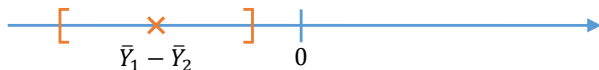


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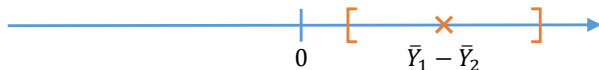


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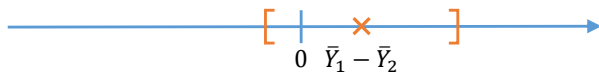
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- Case 2 – Strong evidence that $\theta_1 > \theta_2$:



- Case 3 – No strong evidence that one is larger than the other:



- It does not imply $\theta_1 = \theta_2$!

Comparison of Two Designs

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- We will shrink the CI until case 1 or 2 is achieved, or the confidence interval is so narrow, which suggests that we do not need to separate them.

- For the comparison of performance of two designs, there is an important distinction between
 - *statistically significant difference* (统计意义上的显著区别);
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- However, if the actual difference $|\theta_1 - \theta_2|$ is very small, then it might not be worth the cost to replace design 1 with design 2.
- Confidence intervals do not answer the question of practical significance directly.
 - Instead, they bound, with probability $1 - \alpha$, the true difference $\theta_1 - \theta_2$ within the range $\bar{Y}_1 - \bar{Y}_2 \pm t_{v, 1-\alpha/2} \times \text{s.e.}(\bar{Y}_1 - \bar{Y}_2)$.
 - Whether a difference within these bounds is practically significant depends on the particular problem.

- Independent sampling means that **different** random number streams are used to simulate the two systems.
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$$\text{Var}(\bar{Y}_1 - \bar{Y}_2) = \text{Var}(\bar{Y}_1) + \text{Var}(\bar{Y}_2) = \frac{\sigma_1^2}{R_1} + \frac{\sigma_2^2}{R_2}.$$

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- The $1 - \alpha$ CI is approximated by

$$\bar{Y}_1 - \bar{Y}_2 \pm t_{v, 1-\alpha/2} \times \text{s.e.}(\bar{Y}_1 - \bar{Y}_2). \quad (2)$$

where $\text{s.e.}(\bar{Y}_1 - \bar{Y}_2)$ is given in (1), and the degree of freedom v is

$$v = \frac{[S_1^2/R_1 + S_2^2/R_2]^2}{[S_1^2/R_1]^2/(R_1 - 1) + [S_2^2/R_2]^2/(R_2 - 1)}.$$

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- The approximated CI (2) is called the *Welch confidence interval* (Welch 1938).
 - Sometimes, people will round v to integer for convenience.

- If $R_1 = R_2 = R$, or we are willing to discard some observations from the system design on which we actually have more data, we can pair Y_{r1} with Y_{r2} to define $Z_r = Y_{r1} - Y_{r2}$, for $r = 1, \dots, R$.

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- Approximate $1 - \alpha$ CI:

$$\bar{Z} \pm t_{R-1, 1-\alpha/2} \frac{S}{\sqrt{R}}. \quad (5)$$



- Common Random Numbers (CRN, also known as correlated sampling): For each replication, the same random numbers are used to simulate both systems.
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- $\text{Var}(\bar{Z})$ in (6) is smaller than that in (3) \implies higher precision of point estimator.
- CI is still computed via (4) and (5), but the width will be smaller \implies higher precision.

- It is never enough to simply use the same seed for the random-number generator(s):
 - The random numbers must be synchronized: each random number used in one model for some purpose should be used for the same purpose in the other model.
 - E.g., if the i th random number is used to generate a service time at work station 2 for the 5th arrival in model 1, the i th random number should be used for the very same purpose in model 2.

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- The CRN idea is also used when we validate simulation model via input-output transformation, where we prefer to compare the model and actual system under the same historical input, rather than generate the input from input model.

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- The first three can be achieved by **simultaneous** construction of confidence intervals, whereas the last by some **selection approaches**.
- From now on, without loss of generality, let's *assume the best θ_i is the largest one*.

- Assumption 1: For each design i with mean performance θ_i , the noisy output $Y_{ri} \sim \mathcal{N}(\theta_i, \sigma_i^2)$, for $r = 1, 2, \dots$
- Assumption 2: No CRN is used, i.e., Y_{ri} is independent of Y_{rj} for $i \neq j$.
- Assumption 3 (**indifference-zone**): The gap between the largest θ_i and the second largest θ_i is at least δ , a value known to us.
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- [Bechhofer \(1954\)](#) first developed a selection procedure, which can ensure the probability of correct selection (PCS):

$$\mathbb{P}\{\text{select the largest } \theta_i\} \geq 1 - \alpha, \quad (7)$$

under Assumptions 1-4, where α is a user specified value and $1 - \alpha > 1/k$.

- Bechhofer's Procedure

- 1 Calculate a constant h , which satisfies

$$\mathbb{P}\{Z_i \leq h, i = 1, 2, \dots, k-1\} = 1 - \alpha, \quad (8)$$

where $(Z_1, Z_2, \dots, Z_{k-1})^\top$ has a multivariate normal distribution with means 0, variances 1, and common pairwise correlations $1/2$.

- 2 For $i = 1, \dots, k$, let

$$n_i = \left\lceil \frac{2h^2 \sigma_i^2}{\delta^2} \right\rceil. \quad (9)$$

- 3 For $i = 1, \dots, k$, run n_i replications for design i and calculate

$$\bar{Y}_i = \frac{1}{n_i} \sum_{r=1}^{n_i} Y_{ri}.$$

- 4 Select the design with the largest sample mean \bar{Y}_i as the best.

Proof.

Without loss of generality, assume $\theta_k \geq \theta_{k-1} \geq \dots \geq \theta_1$. Then Assumption 3 says, $\theta_k - \theta_{k-1} \geq \delta$, which implies that

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Now we only need to check that $\mathbf{Z} = (Z_1, Z_2, \dots, Z_{k-1})^\top$ indeed has a multivariate normal distribution with means 0, variances 1, and common pairwise correlations 1/2 (except for some rounding error).

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and $\mathbf{Y} = (\bar{Y}_1, \bar{Y}_2, \dots, \bar{Y}_k)^\top$ is a k -variate normal random vector. So, \mathbf{Z} , as a linear combination of \mathbf{Y} , must be a $(k-1)$ -variate normal random vector.

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For $i \neq j$, $\text{Cov}(Z_i, Z_j) = \text{Cov}\left(\frac{\bar{Y}_i - \bar{Y}_k}{\delta/h}, \frac{\bar{Y}_j - \bar{Y}_k}{\delta/h}\right) = \frac{\text{Cov}(\bar{Y}_i, \bar{Y}_j)}{\delta^2/h^2} = \frac{\sigma_k^2/n_k}{\delta^2/h^2} = \frac{1}{2}.$

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Hence, by (8) and (11), $\mathbb{P}\{\text{select } k\} \geq 1 - \alpha.$



- Assumption 3 (indifference-zone) can be **relaxed** by *softening* the selection target to probability of good selection (PGS):

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 - It requires an initial stage to estimate σ_i^2 by sample variance.
 - The proof is more complicated.
- Procedures like Bechhofer's or Rinott's are simple to implement, but the efficiency may be low.
 - The designed sample size (or, replication number), n_i , may be larger than necessary (too conservative).

- More sample efficient procedures should be in a sequential manner.
 - Take observations sequentially, i.e., one at a time.
 - Eliminate designs from continued sampling when it is statistically clear that they are inferior.
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 - Simulation for a problem with a single dominant alternative may terminate very quickly.
- Paulson (1964) proposed fully sequential procedures, which can guarantee the PCS in (7), under Assumptions 1-3 and (a) *common known* variance or (b) *common unknown* variance.

- Suppose $\sigma_1^2 = \sigma_2^2 = \dots = \sigma_k^2 = \sigma^2$ and σ^2 is known (*common known variance*).
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- Paulson's Procedure

- 1 Let $0 < \lambda < \delta$ (a good choice is $\lambda = \delta/2$), and

$$a = \ln\left(\frac{k-1}{\alpha}\right) \frac{\sigma^2}{\delta - \lambda}.$$

Let $I = \{1, 2, \dots, k\}$ and $r = 0$.

- 2 Let $r \leftarrow r + 1$. Take one observation from each alternative in I and compute $\bar{Y}_i(r)$, $\forall i \in I$.
- 3 Let $I^{\text{old}} = I$ and

$$I = \left\{ \ell \in I^{\text{old}} : \bar{Y}_\ell(r) \geq \max_{i \in I^{\text{old}}} \bar{Y}_i(r) - \max\{0, a/r - \lambda\} \right\}.$$

If $|I| > 1$, then go to Step 2; otherwise, select the alternative left in I as the best.



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- [Kim and Nelson \(2001\)](#) proposed a fully sequential procedure \mathcal{KN} , which extends Paulson's procedure, by allowing *unequal* variances and CRN.
- Commercial simulation software, Simio, implements the \mathcal{KN} procedure of [Kim and Nelson \(2001\)](#) as an Add-In, to help user to select the best scenario.

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 - rank all alternatives
 - select a subset of alternatives
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- Existing procedures for R&S (selection of the best) problems:
 - frequentist
 - Bayesian

- Frequentist procedures typically aim to deliver the PCS or PGS; see Kim and Nelson (2006) for a review:
 - two-stage procedures: Bechhofer (1954), Rinott (1978)
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- Bayesian procedures often allocate samples to each alternative either to maximize the Bayesian posterior PCS or to minimize the expected opportunity cost; see [Chen et al. \(2015\)](#) for a review:
 - optimal computing budget allocation: [Chen et al. \(2000\)](#), [He et al. \(2007\)](#)
 - value of information: [Chick and Inoue \(2001\)](#), [Chick et al. \(2010\)](#)
 - knowledge gradient: [Frazier et al. \(2008\)](#), [Frazier et al. \(2009\)](#)
 - economics of selection procedures: [Chick and Gans \(2009\)](#), [Chick and Frazier \(2012\)](#)

- Emerging research problems that extend classical R&S from different perspectives; see [Hong et al. \(2021\)](#) for a review:
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 - **Simulation Optimization (or called Optimization via Simulation)**

- R&S Problem vs Multi-Arm Bandit (MAB) Problem:

